# Inverse Possibility Analysis Method for Possibility-Based Design Optimization

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**Structural analysis and design optimization have recently been extended to the stochastic approach to consider various uncertainties. However, in areas where it is not possible to produce accurate statistical information for input data, the probabilistic method is not appropriate for stochastic structural analysis and design optimization, because improper modeling of uncertainty could cause a greater degree of statistical uncertainty than those of physical uncertainty. For systems with insufficient information for input data, possibility-based (or fuzzy set) methods have recently been introduced in structural analysis and design optimization. Using possibility methods, the extended fuzzy operations are much simpler than of random variables. Possibility-based design optimization will provide more conservative designs than those from probability methods, and will provide a system level of failure possibility automatically. This paper proposes a new formulation of possibility-based design optimization using the performance measure approach. For the inverse possibility analysis, the maximal possibility search method is proposed to improve numerical efficiency and accuracy comparing with the vertex method and the multilevel-cut method. Two mathematical examples, including a nonmonotonic response and a physical example of vehicle side impact, are used to demonstrate the proposed maximal possibility search method and possibility-based design optimization.**

## Nomenclature

| | | |
|---|---|---|
| $A$ | = | fuzzy event |
| $d$ | = | design vector, $d = [d_1, d_2, \cdots, d_n]^T$ |
| $F_X(x)$ | = | cumulative distribution function of random vector $X$ |
| $G(X)$ | = | performance function; design $X$ is considered "fail" if $G(X) > 0$ |
| $G_{\Pi i}(\bullet)$ | = | $i$th possibilistic constraint using the performance measure approach |
| $i, I$ | = | index and index set |
| $n, nf$ | = | dimension, dimension of fuzzy variables |
| $P(\bullet)$ | = | probability of an event |
| $t$ | = | parametric coordinate along a line in $n$-dimensional cube |
| $V$ | = | vector of standard normalized fuzzy variables with noninteractive triangular membership function; $V = [V_1, V_2, \ldots, V_n]^T$ |
| $v$ | = | realization of $V$; $v = [v_1, v_2, \ldots, v_n]^T$ |
| $v^*$ | = | most possible point |
| $X$ | = | vector of fuzzy variables or random variables; $X = [X_1, X_2, \ldots, X_n]^T$ |
| $x$ | = | realization of $X$; $x = [x_1, x_2, \ldots, x_n]^T$ |
| $Z, z$ | = | random variable uniformly distributed on [0, 1] and its realization |
| $\emptyset$ | = | empty event or empty set |
| $\Omega$ | = | fuzzy event of whole space |
| $\Pi(\bullet)$ | = | possibility of an event |
| $\Pi_X(\bullet)$ | = | membership function of fuzzy variable $X$ |
| $\alpha$ | = | possibility index, which is used for "$\alpha$-cut" |
| $\alpha_t$ | = | target possibility |
| $\lambda$ | = | parameter for definition of convexity |
| $\sigma$ | = | standard deviation of random variable |

## Introduction

STRUCTURAL analysis and design optimization have recently been extended to stochastic approach to consider various uncertainties. Probabilistic methods are prevailing in stochastic structural analysis and design optimization by assuming the amount of raw data is sufficient to create accurate statistical input. However, in many practical applications, sufficient raw data may not be available due to restrictions in time, human and facility resources, and money. In areas where it is not possible to produce accurate statistical information, probabilistic methods are not appropriate for stochastic structural analysis and design optimization, because improper modeling of uncertainty could cause greater degree of statistical uncertainty than those of physical uncertainty [1]. To handle uncertainty with insufficient information, possibility-based (or fuzzy set) methods have recently been introduced in structural analysis and design optimization.

There are two different types of uncertainties: aleatory and epistemic uncertainties [2]. Aleatory uncertainty is classified as objective and irreducible uncertainty, whereas epistemic uncertainty is subjective and reducible uncertainty that stems from lack of knowledge on the input data. A possibility approach is used when dealing with epistemic uncertainty, because this approach is able to deal with epistemic uncertainty by defining a fuzzy variable corresponding to the insufficiently available data [3,4]. The possibility analysis represents a very useful tool to perform operations in the framework of possibility theory [3–9]. The main advantage of the possibility analysis is that it preserves the believable grade of the physical variables through their membership functions. Moreover, extended fuzzy operations are simpler than those required to use probability, especially when a number of variables are

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involved. It has been pointed out that, when little information is available for input data, the possibility-based method is better because it provides a more conservative possibilistic design than the probabilistic design that is consistent with the limited available information [8,10]. This is a desirable merit, because a conservative optimum design is preferred when accurate statistical information is not available.

Unlike the probability-based method in which the probability density function and the cumulative distribution function of the random variable is well known, selection of the membership function for the fuzzy variable in the possibility-based method is not clear. This paper proposes three methods to generate the membership function from the probability distribution function that is not believable by using the probability–possibility consistency principle (see pp. 137, 138 of [11]) or Savoia's formulation [3] for unbelievable data.

As a numerical method for fuzzy analysis, the vertex method is a popular method but could be extremely expensive for large-scale engineering applications. Moreover, the vertex method could yield inaccurate results for fuzzy analysis when the output response is nonmonotonic within the range of input fuzzy variables. Thus, a level-cuts ($\alpha$-cuts) method has been used, where the nonlinear problem was solved with various confidence levels $\alpha$ [8]. Recently, a multilevel-cut method [12] has been developed to overcome the shortcoming of the vertex method for nonlinear possibility analysis, but it is too expensive to carry out possibility-based design optimization (PBDO).

In the probability-based method for structural design, a significant achievement has been made through the performance measure approach (PMA), which transforms probabilistic design formulation into nonprobabilistic one by carrying out an inverse reliability analysis [13,14]. It is found in this paper that PMA is also applicable to the PBDO problem and thus proposed a PMA-based formulation of PBDO to improve numerical efficiency and accuracy. Recently, PMA-based formulation of PBDO was discussed in [15]. This paper formulates the inverse possibility analysis problem for the possibilistic constraint. Unlike the FORM based reliability-based design optimization (RBDO) formulation, which is based on the linear approximation, the PBDO formulation is exact.

Several methods exist in the literature that can be used to solve the PMA-based inverse possibility analysis. Those are the vertex method, the multilevel-cut method, and $\alpha$-level optimization method [16]. As stated before, the vertex method has the accuracy problem, while the multilevel-cut method or $\alpha$-level optimization method is computationally too expensive. To resolve disadvantages of these methods, this paper proposes a new maximal possibility search (MPS) with interpolation method for the inverse possibility analysis, which evaluates possibility constraints efficiently and accurately for nonlinear structural applications.

The proposed MPS method is efficient because it seeks vertex points to obtain the most possible point (MPP). Only when the MPP point is not at the vertex, the interpolation will be used to find the MPP point. Monotonic and nonmonotonic response examples are used to demonstrate the proposed MPS method for the inverse possibility analysis and PMA-based PBDO method.

## Possibility Theories and Definition of Membership Function

### Possibility and Fuzzy Set Theories

This section introduces the basic concepts of the possibility and fuzzy set theories. More information can be found in [10,17]. Possibility is a subjective measure that expresses the degree to which a person considers that an event can occur. The possibility measure $\Pi$ should satisfy the following axioms:

1) Boundary requirement: $\Pi(\emptyset) = 0$, $\Pi(\Omega) = 1$,
2) Monotonicity: if $A_1 \subseteq A_2$, then $\Pi(A_1) \leq \Pi(A_2)$,
3) Union measure: $\Pi(\bigcup_{i \in I} A_i) = \max_{i \in I}\{\Pi(A_i)\}$, $\forall A_i$, $i \in I$, where $\{A_i : i \in I\}$ is a partition of the universal event $\Omega$.

Instead of random variable with statistical distribution function, fuzzy variable with membership function is used for possibility

theory. The membership function of the fuzzy variable represents the "grade of membership." According to Zadeh [17], the membership function $\Pi_X(x)$ should have the properties

1) Bounded: $0 \leq \Pi_X(x) \leq 1$,
2) Possibility of an event: $\Pi(A) = \max_{x \in A}\{\Pi_X(x)\}$.

A fuzzy variable $X$ with the membership function $\Pi_X(x)$ is said to satisfy the *unity* if and only if there exists unique $x$ such that $\Pi_X(x) = 1$. A fuzzy variable $X$ with the membership function $\Pi_X(x)$ is *strongly convex* if and only if the event $\{x | \Pi_X(x) \geq \alpha\}$ is strongly convex $\forall \alpha \in (0, 1]$. An alternative definition of strong convexity is $\Pi_X(\lambda x_1 + (1 - \lambda)x_2) > \min\{\Pi_X(x_1), \Pi_X(x_2)\}$, $\forall x_1, x_2$, and $\forall \lambda \in (0, 1)$. A fuzzy variable $X$ with the membership function $\Pi_X(x)$ is *bounded* if and only if the event $\{x | \Pi_X(x) \geq \alpha\}$ is bounded $\forall \alpha \in (0, 1]$.

The fuzzy variables $X_1$ and $X_2$ are *noninteractive* [10] if the joint membership function $\Pi_{X_1,X_2}(x_1, x_2)$ satisfies $\Pi_{X_1,X_2}(x_1, x_2) = \min\{\Pi_{X_1}(x_1), \Pi_{X_2}(x_2)\}$. The noninteractive assumption in possibility analysis is similar with the independent assumption in probability analysis. This paper assumes that the fuzzy variables are mutually noninteractive. Suppose it is desired to estimate the possibility of some output failure events, where the joint membership function of two input variables is not known, but the marginal membership functions of these variables are available. If we assume these variables are noninteractive, it will yield the maximum possibility of the output failure event compared with the case that the joint membership function of these variables is used. That is, the assumption of noninteractive variables will yield a more conservative design in PBDO.

### Generation of the Input Membership Functions

In most engineering design optimization applications, the input uncertainties could be aleatory by nature. However, when it is not possible to obtain sufficient data due to lack of resources (labor, time, overhead, etc.), treating the input uncertainty as aleatory (i.e., random variable) will be improper, which in turn will lead to an erroneous reliability-based optimum design. In this case, it is better to treat the input uncertainty as epistemic (i.e., fuzzy variable) and use PBDO.

The generation of proper input membership functions of the fuzzy variables using the available limited set of data is an important step of the inverse possibility analysis and successful PBDO. There are several methods in literature to generate membership function from input data [18–23]. However, some membership functions generated by these methods cannot be used for the PBDO method proposed in this paper. That is, the input membership functions must satisfy the unit, strongly convexity, and boundedness properties described in previous section.

If the available probability density function of the input variable is not believable to be used for RBDO, the input variable is treated as fuzzy variable and the corresponding membership function can be generated from the probability density function. There are three methods proposed in this paper. The probability–possibility consistency principle and the least conservative principle [10,11] are used to generate the membership function from the probability density function. The probability–possibility consistency principle states that, the possibility of one event cannot be less than the probability of the same event. The membership function that satisfies the probability–possibility consistency principle is not unique. Thus, for the design purpose, the least conservative membership function should be chosen such that it does not yield a design that is too conservative.

For the first method, if the cumulative distribution function of the input variable is $F_X(x)$, the membership function of the fuzzy variable that satisfies the probability–possibility consistency principle and the least conservative principle is

$$\Pi_X(x) = 1 - |2F_X(x) - 1|$$
$$= \begin{cases} 2F_X(x), & x \in \{x : F_X(x) \leq 0.5\} \\ 2 - 2F_X(x), & x \in \{x : F_X(x) > 0.5\} \end{cases} \quad (1)$$

which is unique.

To show this, if the cumulative distribution function of the input variable $Z$ is $F_Z(z) = z$ where $0 \leq z \leq 1$, which means the distribution is uniformly distributed on [0, 1], the membership function of this fuzzy variable can be assumed to be symmetric on [0,1]. Now, if $z \in [0, 0.5]$, by letting $A_z = [0, z] \cup [1 - z, 1]$, we have $\Pi_Z(z) = \Pi_Z(1 - z) = \Pi(A_z)$ using the symmetry. On the other hand, we have $P(A_z) = 2z = 1 - |2z - 1|$, where the second equality is for the case of $z \in [0.5, 1]$. The probability–possibility consistency principle [11] implies that $\Pi(A_z) \geq P(A_z)$. Now using the least conservative principle, the triangular membership function of the fuzzy variable $Z$ can be formulated as $\Pi_Z(z) = 1 - |2z - 1|$. Finally, for the general case, using the transformation $Z = F_X(X)$, any input variable can be transferred to a fuzzy variable with uniform distribution function, which yields the unique membership function in Eq. (1).

For the second method, for a design with the system level target possibility, the probability–possibility consistency principle [10] should be used for the entire $nf$-dimensional space, where $nf$ is the number of fuzzy variables. Thus, the membership function of the fuzzy variable should be

$$\Pi_X(x) = 1 - |2F_X(x) - 1|^{nf} \tag{2}$$

To show this, assume that the CDFs of the $nf$ input variables are independent, and thus the corresponding fuzzy variables are noninteractive. Next transfer all CDFs into uniform distributions on [0, 1] and construct the event $A_z = [0, 1]^{nf} \backslash [z, 1 - z]^{nf}$ where $z \in [0, 0.5]$. Then we can obtain $P(A_z) = 1 - |2z - 1|^{nf}$ using the independency. Next, using the probability–possibility consistency principle and the least conservative principle, we have $\Pi(A_z) = 1 - |2z - 1|^{nf}$. The symmetry and the noninteractive assumption provide $\Pi_Z(z) = 1 - |2z - 1|^{nf}$, which yields Eq. (2) by transferring back to the variable $X$.

For the third method, if the membership functions generated using these two principles are not viewed as conservative enough, Savoia ([3], p. 1096) proposed the membership function as

$$\Pi_X(x) = F_X(x_L) + f_X(x)(x_R - x_L) + F_X(x_R) \tag{3}$$

where $f_X(x)$ is the probability density function, and $x_L$ and $x_R$ are selected such that $x_L \leq x_R$ and $f_X(x_L) = f_X(x_R) = f_X(x)$.

## Possibility-Based Design Optimization

### Formulation of Possibility-Based Design Optimization

For general engineering applications, a possibility-based design optimization (PBDO) model can be formulated to

minimize $\text{cost}(\mathbf{d})$

subject to $\Pi(G_i[\mathbf{X}(\mathbf{d})] > 0) \leq \alpha_t,$ $\quad i = 1, 2, \ldots, nc$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \tag{4}$$

where $\mathbf{d} = [d_i^T] \in R^n$ is the design vector; $\mathbf{X} = [X_i]^T \in R^{nf}$ is the vector of variables where each fuzzy variable $X_i$ has the membership function $\Pi_{X_i}(x_i)$ and the maximal grade [17] $\max\{\Pi_{X_i}(x_i)\} = d_i$; $\alpha_t$ is the target possibility of failure; and $n$, $nf$, and $nc$ are the number of design variables, fuzzy variables, and possibility constraints, respectively. The possibility constraints are described by $\Pi(\bullet) \leq \alpha_t$ for any failure event $G_i[\mathbf{X}(\mathbf{d})] > 0$. In this paper only the series system is considered, which means that the system fails if any of its components fail.

### Performance Measure Approach in Possibility-Based Design Optimization

The performance measure approach (PMA) has been successfully applied to reliability-based design optimization (RBDO), significantly improving numerical stability and efficiency [13,14]. The PMA approach, which was developed for the reliability-based design problem, instead of the reliability analysis, is applicable to the PBDO model to

minimize $\text{cost}(\mathbf{d})$

subject to $G_{\Pi_i}[\mathbf{X}(\mathbf{d})] \leq 0,$ $\quad i = 1, 2, \ldots, nc$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \tag{5}$$

where $G_{\Pi_i}$ is $i$th possibility constraint, which is the maximum value of the constraint on the $\alpha_t$ cut. The equivalence of PMA approach in (2) and the original formulation in (1) is clear because both $\Pi(G_i[\mathbf{X}(\mathbf{d})] > 0) \leq \alpha_t$ and $G_{\Pi_i}[\mathbf{X}(\mathbf{d})] \leq 0$ imply that the constraint is feasible on the $\alpha_t$ cut.

In this paper, it is assumed that the noninteractive input fuzzy variables $X_i$ have its membership function $\Pi_{X_i}(x_i)$ satisfying three properties [3,10,17]: 1) unity, 2) strong convexity, and 3) boundedness. These three properties make it possible for the noninteractive input fuzzy variables $X_i$, $i = 1, \ldots, nf$ to be uniquely transformed to the standard normalized fuzzy variables $V_i$, $i = 1, \ldots, nf$ with noninteractive isosceles triangular membership functions as

$$\Pi_{V_i}(v_i) = \left\{ \begin{array}{ll} v_i + 1, & -1 \leq v_i \leq 0 \\ 1 - v_i, & 0 \leq v_i \leq 1 \end{array} \right\} = 1 - |v_i|, \quad |v_i| \leq 1$$
$$i = 1, \ldots, nf \tag{6}$$

The transformation can be written as

$$V_i = \left\{ \begin{array}{ll} \Pi_{X_i,L}(X_i) - 1, & X_i \leq d_i \\ 1 - \Pi_{X_i,R}(X_i), & X_i > d_i \end{array} \right. \tag{7}$$

where $\Pi_{X_i,L}(x_i)$ and $\Pi_{X_i,R}(x_i)$ are the left side and right side of the membership function of the input fuzzy variable $X_i$, respectively, and $d_i$ is the maximal grade of this membership function.

Thus, evaluation of the possibility constraint requires an inverse possibility analysis using PMA, which is formulated to

maximize $G(V)$ $\quad$ subject to $||V||_\infty \leq 1 - \alpha_t \tag{8}$

where the optimum point on the target possibility domain $||V||_\infty \leq 1 - \alpha_t$ is identified as the MPP $\mathbf{u}_{\alpha_t}^*$ with the prescribed possibility of failure $\alpha_t$. Here $||V||_\infty$ is the supremum norm of the vector $\mathbf{V}$. Notice that the target possibility domain $||V||_\infty \leq 1 - \alpha_t$ is the same as the $\alpha_t$ cut.

The difference between the inverse reliability analysis [13,14] and the inverse possibility analysis [3–8,10,12,16,17] is that the most probable point (MPP) in the inverse reliability analysis is based on FORM, which means the related probability is first order approximation, whereas the most possible point (MPP) in the inverse possibility analysis is exact, along with the related possibility. Another difference is the search domain enclosed by the target confidence level. The inverse reliability analysis has $nr$-dimensional hypersphere as its search domain $||U||_2 \leq \beta_t$, whereas the inverse possibility analysis has $nf$-dimensional hypercube as its search domain, $||V||_\infty \leq 1 - \alpha_t$, that makes the numerical computation simpler, compared with the inverse reliability analysis. Because the MPP search in the inverse possibility analysis is different from the one in the inverse reliability analysis, a new numerical method is proposed in the next section to solve the inverse possibility analysis problem in Eq. (8).

## Maximal Possibility Search Method for Inverse Possibility Analysis

For the inverse possibility analysis of the nonmonotonic response within the range of input fuzzy parameters, the vertex method [3] could be inaccurate; and the multilevel-cut method or other computational methods such as a global optimization method [16] could be computationally expensive. In this paper, the maximal possibility search (MPS) method is proposed for the inverse possibility analysis to ensure numerical efficiency and accuracy in PBDO. This method first attempts to find an MPP using the proposed maximal possibility search, because in majority of cases the MPP is likely to be at a vertex of the target possibility domain

$||V||_\infty \leq 1 - \alpha_t$. If the proposed maximal possibility search does not yield a solution at a vertex, then the maximal possibility search with an interpolation is used to search MPP on the edge or in the interior domain of the hypercube.

**Maximal Possibility Search**

The main idea of the MPS algorithm is based on the fact that MPP is likely be on the vertex point because the search domain of the inverse possibility analysis is a hypercube. Using the constraint gradients at the current point of current iteration, the new vertex point is searched by selecting the end of the $\alpha_t$-cut interval of each variable according to the sign of the gradient with respect to each variable. If the constraint value at this new vertex point is larger than at the current vertex point, pick the new point as the current point and repeat. The convergence criteria for the vertex point are that, the gradient vector is pointing out of MPP search domain for each variable. If the constraint at the new vertex point is smaller than at the current point, then use an interpolation in order to obtain the new point that has a larger constraint value.

The proposed maximal possibility search is as following:

Step 1: Set the iteration counter $k = 0$ with the convergence parameter $\varepsilon = 10^{-3}$. Set $j = 1$. Let $\mathbf{v}^{(0)} = \mathbf{0}$. Calculate the performance $G(\mathbf{v}^{(0)})$ and the sensitivity $\nabla G(\mathbf{v}^{(0)})$. Let $\mathrm{d}^{(k)} = \nabla G(\mathbf{v}^{(k)})$.

Step 2: Search the next point as $\mathbf{v}^{(k+1)} = \pi_t \cdot \mathrm{sgn}(\mathrm{d}^{(k)})$ where $\pi_t = 1 - \alpha_t$ and $\mathrm{sgn}(X) = [\mathrm{sgn}(X_1), \mathrm{sgn}(X_2), \cdots, \mathrm{sgn}(X_{nf})]$ if $X = [X_i]^T \in R^{nf}$. Let $k = k + 1$.

Step 3: Calculate the performance $G(\mathbf{v}^{(k)})$ and its sensitivity $\nabla G(\mathbf{v}^{(k)})$. Let $\mathrm{d}^{(k)} = \nabla G(\mathbf{v}^{(k)}) + \beta \mathrm{d}^{(k-1)}$ where $\beta = [||\nabla G(\mathbf{v}^{(k)})||/||\nabla G(\mathbf{v}^{(k-1)})||]^2$. If $\mathrm{sgn}[\nabla G(\mathbf{v}^{(k)})] = \mathrm{sgn}(\mathbf{v}^{(k)})$, it is the maximum point and stop. If $G(\mathbf{v}^{(k)}) \geq G(\mathbf{v}^{(j)})$, let $j = k$ and go to step 2. Otherwise, go to step 4.

Step 4: Go to step 5, with $\mathbf{v}^{(j)}$, $G(\mathbf{v}^{(j)})$ and $\nabla G(\mathbf{v}^{(j)})$.

**Maximal Possibility Search with an Interpolation**

The interpolation is used between the current point and the point on the boundary identified from the current point along the ascent direction, rather than the vertex point. Because this algorithm is sensitivity based, a third order polynomial approximation of the performance function is used. Using the interpolation, a new point that maximizes the constraint value is selected. The convergence criteria for the nonvertex maximum point are that, for the point on the boundary, the gradient should be pointing out of the interval, whereas for components of nonend point, the gradient should vanish.

The proposed maximal possibility search with an interpolation is as follows:

Step 5: Let $l = 0$ and $\mathrm{d}^{(l)} = \nabla G(\mathbf{v}^{(j)})$. Go to step 6.

Step 6: Calculate the new point $\mathbf{v}^{(k+1)}$ on the boundary of the domain from the start point $\mathbf{v}^{(j)}$ along the search direction $\mathrm{d}^{(l)}$. Let $k = k + 1$.

Step 7: Calculate the performance $G(\mathbf{v}^{(k)})$ and its sensitivity $\nabla G(\mathbf{v}^{(k)})$. If

$$\begin{cases} \mathrm{sgn}\left(\frac{\partial G}{\partial x_i}(\mathbf{v}^{(k)})\right) = \mathrm{sgn}(v_i), & \text{for } v_i = \pi_t \text{ or } v_i = -\pi_t \\ \left|\frac{\partial G}{\partial x_i}(\mathbf{v}^{(k)})\right| < \varepsilon, & \text{for } -\pi_t < v_i < \pi_t \end{cases} \quad (9)$$

then it is the maximum point and stop. Otherwise, go to step 8.

Step 8: Use $G(\mathbf{v}^{(j)})$, $G(\mathbf{v}^{(k)})$, $\nabla G(\mathbf{v}^{(j)})$ and $\nabla G(\mathbf{v}^{(k)})$ to construct the third order polynomial $P_G(t)$ on the straight line between $\mathbf{v}^{(j)}$, and $\mathbf{v}^{(k)}$ where $t$ is the parameter for the line. Calculate the maximum point $t^*$ for this polynomial. Let $\mathbf{v}^{(k+1)}$ be the point on the line corresponding to $t^*$. Let $k = k + 1$.

Step 9: Calculate the performance $G(\mathbf{v}^{(k)})$ and the sensitivity $\nabla G(\mathbf{v}^{(k)})$. Check the convergent criteria using the equation in Step 7. If convergent, stop. Otherwise, let the new conjugate direction be $\mathrm{d}^{(l+1)} = \nabla G(\mathbf{v}^{(k)}) + \beta \mathrm{d}^{(l)}$ where $\beta = (||\nabla G(\mathbf{v}^{(k)})||/||\nabla G(\mathbf{v}^{(k-2)})||)^2$. Let $j = k$, $l = l + 1$, and go to step 6.

*Remark*: The proposed maximal possibility search is sufficient for the monotonic responses because the MPP is likely to be at a vertex. The proposed maximal possibility search with an interpolation will be used only when the maximal possibility search fails. Because in most cases the maximal possibility search finds MPP efficiently, the MPS method will be efficient while robust for nonlinear and nonmonotonic responses.

## Numerical Examples

**Inverse Possibility Analysis of Mathematical Problems**

For the first example, consider the input fuzzy variables $X_1$ and $X_2$ that are noninteractive and have the same triangular membership function on the interval [4.737, 7.263]. The performance function is

$$G_1(X) = \exp(X_1 - 7) + X_2 - 10 \quad (10)$$

The target possibility $\alpha_t$ is set to 0.05, which is equivalent to the possibility of failure equal to 0.05 and the design point is $d = [6.0, 6.0]^T$. Using the proposed MPS method, the MPP is obtained in two steps, as shown in Table 1 and in Fig. 1a. In Table 1 and the tables that are following, $V_i$s are the standard normalized fuzzy variables that are transformed from $X_i$s as described in Eq. (6) and (7).

As a second example, consider the input fuzzy variables $X_1$ and $X_2$ that are noninteractive and have the triangular membership functions on the intervals (2.737, 5.263) and (3.737, 6.263), respectively. The performance function is

$$G_2(X) = -[\exp(0.8X_1 - 1.2) + \exp(0.7X_2 - 0.6) - 5]/10 \quad (11)$$

The target possibility $\alpha_t$ is set to 0.05 and the design point is $d = [4.0, 5.0]^T$. Using the proposed maximal possibility search, the solution is again obtained in one iteration, as shown in Table 2 and in Fig. 1b.

For the third example, assume noninteractive input fuzzy variables $X_1$ and $X_2$ have the triangular membership functions on the interval $|X_i^U - X_i^L| = 1.737$, $i = 1, 2$. A midpoint of the interval corresponds to the design point. The performance function is given as

$$G_3(X) = -0.3X_1^2 X_2 + X_2 - 0.8X_1 - 1 \quad (12)$$

The target possibility $\alpha_t$ is set to 0.05 and the design point is $d = [-0.5, 2.2]^T$. The proposed maximal possibility search fails, but the proposed maximal possibility search with an interpolation gives the solution in three iterations as shown in Table 3 and in Fig. 2a.

**Table 1  Inverse possibility analysis of $G_1(X)$**

| Iteration | $V_1$ | $V_2$ | $G_1$ | $\partial G_1/\partial V_1$ | $\partial G_1/\partial V_2$ |
|---|---|---|---|---|---|
| 0 | 0.00 | 0.00 | −3.632 | 0.465 | 1.263 |
| 1 | 0.95 | 0.95 | −1.579 | 1.543 | 1.263 |
| MPP | 0.95 | 0.95 | −1.579 | 1.543 | 1.263 |



a) $G_1(X)$    b) $G_2(X)$

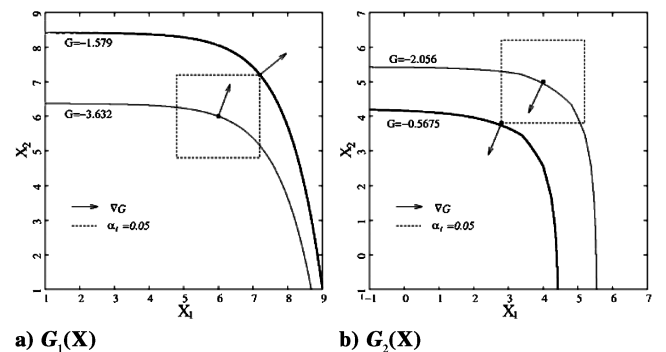**Fig. 1  Inverse possibility analysis history of monotonic responses.**

**Table 2  Inverse possibility analysis of $G_2(X)$**

| Iteration | $V_1$ | $V_2$ | $G_1$ | $\partial G_1/\partial V_1$ | $\partial G_2/\partial V_2$ |
|---|---|---|---|---|---|
| 0 | 0.00 | 0.00 | −2.056 | −0.747 | −1.607 |
| 1 | −0.95 | −0.95 | −0.568 | −0.286 | −0.694 |
| MPP | −0.95 | −0.95 | −1.568 | −0.286 | −0.694 |

**Table 3  Inverse possibility analysis of nonmonotonic $G_3(X)$ at $d = [-0.5, 2.2]^T$**

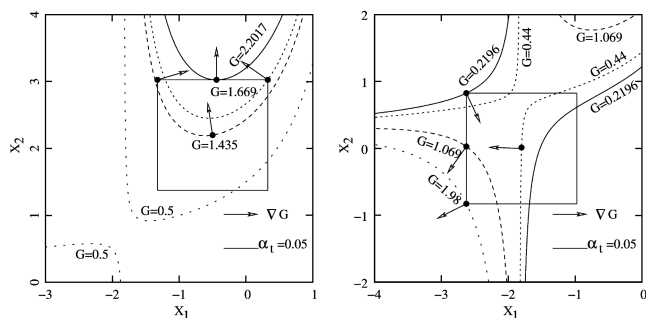| Iteration | $V_1$ | $V_2$ | $G_3$ | $\partial G_3/\partial V_1$ | $\partial G_3/\partial V_2$ |
|---|---|---|---|---|---|
| 0 | 0.00 | 0.00 | 1.435 | −0.122 | 0.803 |
| 1 | −0.95 | 0.95 | 1.492 | 1.394 | 0.411 |
| 2 | 0.95 | 0.95 | 1.669 | −1.207 | 0.841 |
| 3[a] | 0.068 | 0.95 | 2.201 | 0.000 | 0.818 |
| MPP | 0.068 | 0.95 | 2.201 | 0.000 | 0.818 |

[a]Maximal possibility search with an interpolation.

For the fourth example, consider the performance function in Eq. (12). With the same target possibility $\alpha_t = 0.05$ and the design point $d = [-1.8, 0.0]^T$, the nonmonotonic response becomes a saddle response. The proposed MPS method finds the solution in three iterations, as shown in Table 4 and in Fig. 2b.

**Table 4  Inverse possibility analysis of nonmonotonic $G_3(X)$ at $d = [-1.8, 0.0]^T$**

| Iteration | $V_1$ | $V_2$ | $G_3$ | $\partial G_3/\partial V_1$ | $\partial G_3/\partial V_2$ |
|---|---|---|---|---|---|
| 0 | 0.00 | 0.00 | 0.440 | −0.695 | 0.024 |
| 1 | −0.95 | 0.95 | 0.220 | 0.434 | −0.927 |
| 2[a] | −0.95 | −0.95 | 1.980 | −1.823 | −0.927 |
| 3[a] | −0.95 | −0.95 | 1.980 | −1.823 | −0.927 |
| MPP | −0.95 | −0.95 | 1.980 | −1.823 | −0.927 |

[a]Maximal possibility search with an interpolation.

**Table 5  Inverse possibility analysis of side impact problem**

| Iter. | $V_3$ | $V_5$ | $V_6$ | $V_7$ | $V_9$ | $V_{10}$ | $V_{11}$ | $G_9$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.018 |
|  | − | − | − | − | 0 | + | − |  |
| 1 | −0.95 | −0.95 | −0.95 | −0.95 | 0.0 | 0.95 | −0.95 | 0.421 |
|  | − | − | − | − | + | + | + |  |
| 2 | −0.95 | −0.95 | −0.95 | −0.95 | 0.95 | 0.95 | −0.95 | 0.437 |
|  | − | − | − | − | + | + | + |  |
| 3 | −0.95 | −0.95 | −0.95 | −0.95 | 0.95 | 0.95 | 0.95 | 0.130 |
|  | − | − | − | − | − | + | − |  |
| 4[a] | −0.95 | −0.95 | −0.95 | −0.95 | 0.95 | 0.95 | −0.63 | 0.461 |
|  | − | − | − | − | + | + | 0 |  |
| MPP | −0.95 | −0.95 | −0.95 | −0.95 | 0.95 | 0.95 | −0.63 | 0.461 |

[a]Maximal possibility search with an interpolation.

### Inverse Possibility Analysis of a Vehicle Side Impact Problem

Assume all fuzzy variables are noninteractive and fuzzy variables $X_1 \sim X_7$ have the same triangular membership functions on the interval (0.91, 1.09); $X_8$ and $X_9$ have the same triangular membership functions on the interval (0.282, 0.318); and $X_{10}$ and $X_{11}$ have the same triangular membership functions on the interval (−30.0, 30.0). From the example of vehicle side impact, the limit state function of upper rib deflection is defined as [24]

$$G(X) = 1.35 - 0.489X_3X_7 - 0.843X_5X_6 + 0.0432X_9X_{10}$$
$$- 0.0556X_9X_{11} - 0.000786X_{11}^2 \qquad (13)$$

The target possibility $\alpha_t$ is set to 0.05. Using the proposed MPS method, the solution is obtained in four iterations, as shown in Table 5. There are two lines for each iteration: the first line is the components of the point in standard $V$ space and the performance function evaluated at this point, while second line is the sign of the gradient of the performance function for each component. It is confirmed that the search method finds the correct MPP that has a local maximum of $G(V)$ in Eq. (8), as shown in Table 5, because the sign of the components of the point in $V$ space coincide with the sign of the component of the gradient, except the 11th constraint, of which sensitivity vanishes because the MPP point is not on the vertex.

### Mathematical Design Problem

Consider the following mathematical design problem to

minimize $d_1 + d_2$

subject to $G_1(X) = 1 - X_1^2 X_2/20 \le 0$

$G_2(X) = 1 - (X_1 + X_2 - 5)^2/30 - (X_1 - X_2 - 12)^2/120 \le 0$

$G_3(X) = 1 - 80/(X_1^2 + 8X_2 + 5) \le 0$

$d^L \le d \le d^U \qquad (14)$

where $d^L = [0, 0]^T$ and $d^U = [10, 10]^T$. For this problem, suppose that true randomness of design variables $X_1$ and $X_2$ are normally distributed with the standard deviation $\sigma = 0.3$. The objective is to obtain an optimum design that yields failure rate less than 0.02275. However, imagine that, due to lack of information, we do not know the correct standard deviation of these variables. At the initial design $X_1 = X_2 = 5$, statistical sample has been taken to generate a histogram shown in Fig. 3. From the sample, the standard deviation is
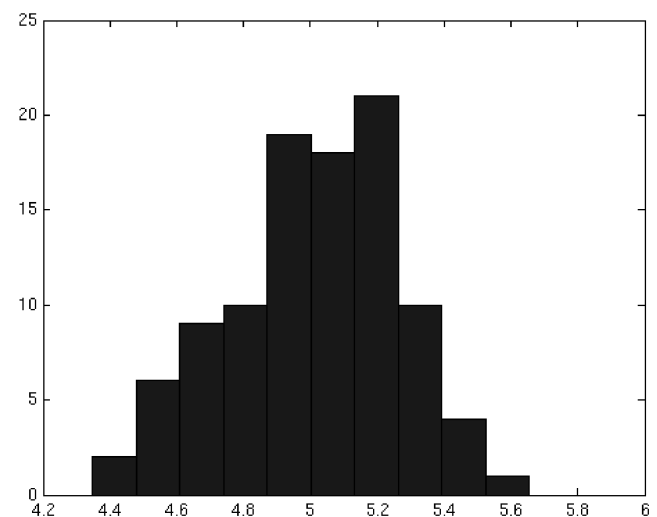


**Fig. 3  Histogram of the uncertainties for mathematical design problem.**



a) $G_3(X)$ at $d = [-0.5, 2.2]^T$    b) $G_3(X)$ at $d = [-1.8, 0.0]^T$

**Fig. 2  Inverse possibility analysis history of nonmonotonic responses.**

**Table 6 Possibility-based design optimization history using the performance measure approach for the mathematical design problem**

| Iteration | cost | $d_1$ | $d_2$ | $G_1$ | $G_2$ | $G_3$ | No. of Analyses |
|---|---|---|---|---|---|---|---|
| 0 | 10.00 | 5.000 | 5.000 | −3.287 | −0.686 | −0.012 | 3(2/0/1) |
| 1 | 7.615 | 4.004 | 3.610 | −0.760 | −0.134 | −0.340 | 7(2/4/1) |
| 2 | 6.710 | 3.485 | 3.225 | −0.104 | −0.027 | −0.535 | 4(2/1/1) |
| 3 | 6.525 | 3.408 | 3.117 | −0.003 | −0.001 | −0.580 | 4(2/1/1) |
| Opt. | 6.525 | 3.408 | 3.117 | Act. | Act. | Inact. | 18(8/6/4) |

**Table 7 Reliability-based design optimization history of mathematical design problem**

| Iteration | cost | $d_1$ | $d_2$ | $G_1$ | $G_2$ | $G_3$ | No. of Analyses |
|---|---|---|---|---|---|---|---|
| 0 | 10.00 | 5.000 | 5.000 | −3.906 | −0.762 | −0.042 | 6(2/2/2) |
| 1 | 7.497 | 3.826 | 3.671 | −0.944 | −0.197 | −0.458 | 7(3/2/2) |
| 2 | 6.333 | 3.342 | 2.991 | −0.143 | −0.040 | −0.754 | 7(3/2/2) |
| 3 | 6.061 | 3.262 | 2.799 | −0.005 | −0.002 | −0.839 | 7(3/2/2) |
| Opt. | 6.061 | 3.262 | 2.799 | Act. | Act. | Inact. | 27(11/8/8) |

**Table 8 Probabilities of failure for components and system**

| Optimum Designs | cost | $d_1$ | $d_2$ | $P(G_1 > 0)$ | $P(G_2 > 0)$ | $P(G_3 > 0)$ | $P$(System Failure) |
|---|---|---|---|---|---|---|---|
| PBDO | 6.525 | 3.408 | 3.117 | 0.00414 | 0.00501 | 0.00000 | 0.00913 |
| RBDO | 6.061 | 3.262 | 2.799 | 0.04356 | 0.03393 | 0.00000 | 0.07610 |

estimated to be 0.2592. For this problem, both PBDO and RBDO are carried out to compare the optimum results. For PBDO, the membership functions are generated using the first method in Sec. 2.2.2 from the approximate normal distribution with standard deviation 0.2592. For RBDO, the estimated input normal distributions are used.

The PBDO model is to

$$\text{minimize } d_1 + d_2$$
$$\text{subject to } \Pi(G_i[X(d)] > 0) \le \alpha_{t_i}, \qquad i = 1, 2, 3$$
$$d^L \le d \le d^U \qquad (15)$$

where $\alpha_{t_1} = \alpha_{t_2} = 0.02275$. The initial design is $d^{(0)} = [5.0, 5.0]^T$.

Using PMA, the PBDO model can be reformulated as to

$$\text{minimize } d_1 + d_2$$
$$\text{subject to } G_{\Pi i}[X(d)] \le 0, \qquad i = 1, 2, 3$$
$$d^L \le d \le d^U \qquad (16)$$

where $G_{\Pi i}[X(d)]$ are calculated by carrying out the inverse possibility analysis to

$$\text{maximize } G_i(V) \qquad \text{subject to } ||V||_\infty \le 1 - \alpha_{t_i} \qquad (17)$$

in the standard normalized $V$ space.

Using a sequential quadratic programming (SQP) [25], the PBDO history is presented in Table 6 with the result shown in Fig. 4a graphically. Inside the parenthesis of the last column "No. of analyses" of Table 6 corresponds to the number of analyses required for each constraints $G_i$, $i = 1 \sim 3$.

For comparison, the corresponding RBDO model is to

$$\text{minimize } d_1 + d_2$$
$$\text{subject to } P(G_i[X(d)] > 0) \ge \Phi(-\beta_{t_i}), \qquad i = 1, 2, 3$$
$$d^L \le d \le d^U \qquad (18)$$

where $\beta_{t_1} = \beta_{t_2} = 2.0$, which correspond to the component target probability of failure of $\Phi(-\beta_{t_1}) = \Phi(-\beta_{t_2}) = 0.02275$. The estimated input normal distributions are used.

Using SQP optimizer, the RBDO history is presented in Table 7 with the result shown in Fig. 4b graphically. Note that the optimum cost of the PBDO result shown in Table 6 is larger than that of the RBDO result shown in Table 7.

To compare these optimum designs, we can use the true normal distributions of $X_1$ and $X_2$ that have standard deviation $\sigma = 0.3$. The probability of failure of the RBDO and PBDO designs for each component $G_i(X; d)$ and the system are shown in the Table 8. The system level probability of failure is obtained using the Monte Carlo simulation for each optimum design.

From Table 8, it can be seen that the PBDO result is more conservative than the RBDO result. Note that, for both PBDO and RBDO models, the target probability of failure and target possibility of failure rate was set to be 0.02275. An important property of PBDO is that the optimum design satisfies the system level failure target. That is, the PBDO result yields the system level failure of 0.00913, which is less than 0.02275. On the other hand, the RBDO result, obtained using the approximate statistical distributions, does not



a) PBDO result      b) RBDO result
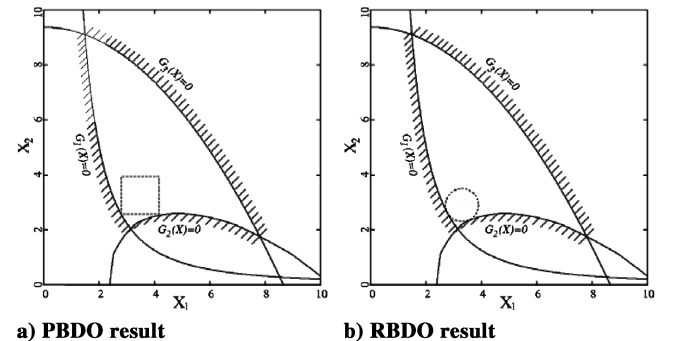
**Fig. 4 Possibility-based design optimization and reliability-based design optimizatio results of mathematical design problem.**

even satisfy the component level failure target. That is, first two constraints and the system do not satisfy the target probability of failure of 0.02275.

## Vehicle Side Impact Design Problem

A vehicle side impact design problem shown in Fig. 5 is described in [24] in detail. The design problem is to minimize the vehicle weight while enhancing the side impact crash performance. That is, find $\boldsymbol{d} = [d_1, \ldots, d_9]$ to

minimize

$$\text{cost}(\boldsymbol{d}) = 1.98 + 4.9d_1 + 6.67d_2 + 6.98d_3$$
$$+ 4.01d_4 + 1.78d_5 + 2.73d_7$$

subject to

$$G_1(\boldsymbol{X}) = 14.36 + (-9.9X_2 - 12.9X_1X_8 + 0.110X_3X_{10} \le 0$$

$$G_2(\boldsymbol{X}) = 1.86 + (2.95X_3 + 0.1792X_{10} - 5.057X_1X_2$$
$$- 11X_2X_8 - 0.0215X_5X_{10} - 9.98X_7X_8 + 22X_8X_9) \le 0$$

$$G_3(\boldsymbol{X}) = -3.02 + (3.818X_3 - 4.2X_1X_2 + 0.0207X_5X_{10}$$
$$+ 6.63X_6X_9 - 7.7X_7X_8 + 0.32X_9X_{10}) \le 0$$

$$G_4(\boldsymbol{X}) = -0.059 + (-0.0159X_1X_2 - 0.188X_1X_8$$
$$- 0.019X_2X_7 + 0.0144X_3X_5 + 0.0008757X_5X_{10}$$
$$+ 0.08045X_6X_9 + 0.00139X_8X_{11} + 0.00001575X_{10}X_{11}) \le 0$$

$$G_5(\boldsymbol{X}) = -0.106 + (0.00817X_5 - 0.131X_1X_8 - 0.070X_1X_9$$
$$+ 0.03099X_2X_6 - 0.018X_2X_7 + 0.0208X_3X_8 + 0.121X_3X_9$$
$$- 0.00364X_5X_{10} - 0.0005354X_6X_{10} + 0.0012X_8X_{11}$$
$$+ 0.00184X_9X_{10} - 0.018X_2^2) \le 0$$

$$G_6(\boldsymbol{X}) = 0.42 + (-0.61X_2 - 0.163X_3X_8 + 0.001232X_3X_{10}$$
$$- 0.166X_7X_9 + 0.227X_2^2) \le 0$$

$$G_7(\boldsymbol{X}) = 0.72 + (-0.5X_4 - 0.19X_2X_3 - 0.0122X_4X_{10}$$
$$+ 0.009325X_6X_{10} + 0.00019X_{11}^2) \le 0$$

$$G_8(\boldsymbol{X}) = 0.68 + (-0.674X_1X_2 - 1.95X_2X_8 + 0.02054X_3X_{10}$$
$$- 0.0198X_4X_{10} + 0.028X_6X_{10}) \le 0$$

$$G_9(\boldsymbol{X}) = 1.35 + (-0.489X_3X_7 - 0.843X_5X_6 + 0.0432X_9X_{10}$$
$$- 0.0556X_9X_{10} - 0.0556X_9X_{11} - 0.000786X_{11}^2) \le 0$$

$$G_{10}(\boldsymbol{X}) = 0.16 + (-0.3717X_2X_4 - 0.00931X_2X_{10}$$
$$- 0.484X_3X_9 + 0.01343X_6X_{10}) \le 0$$

and $\quad \boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U$ (19)

Suppose true input uncertainties are all normally distributed, with standard deviations of 0.05 for $X_1 \sim X_7$; 0.006 for $X_8$, $X_9$; and 10.0 for $X_{10}$, $X_{11}$, respectively. The objective is to obtain an optimum design that yields failure rate less than 0.02275. However, again, due to lack of information, assume that we do not know the correct standard deviation of these variables.
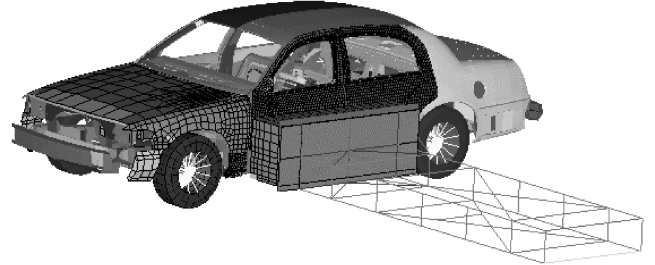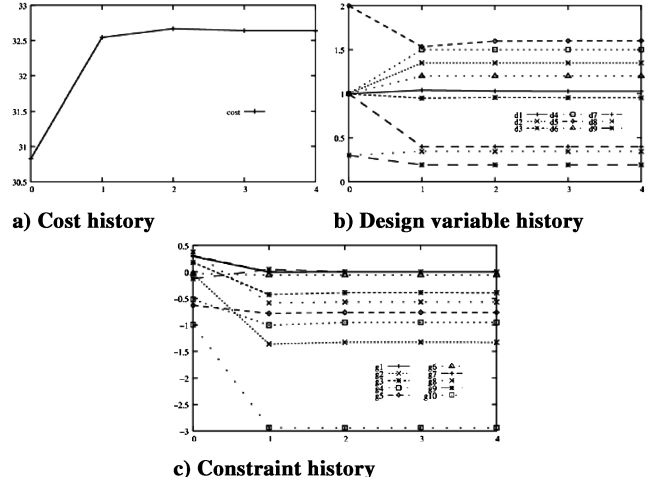


**Fig. 5   Side impact model.**



**a) Cost history**          **b) Design variable history**



**c) Constraint history**

**Fig. 6   Possibility-based design optimization design history of side impact problem.**

The PBDO model for vehicle impact design problem is to

minimize $\text{cost}(\boldsymbol{d})$      subject to $\Pi(G_i[\boldsymbol{X}(\boldsymbol{d})] > 0) \ge \alpha_t$,
$$i = 1, \ldots, 10 \qquad \boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U \qquad (20)$$

where the initial design is $d_i = 1$, $i = 1 \sim 4, 6, 7$, $d_5 = 2$, and $d_8 = d_9 = 0.03$.

When there is lack of data information, for RBDO, design engineers often use the uniform distributions with certain lower and upper bound, because it is believed that the uniform input distributions provide a conservative optimum design. On the other hand, for PBDO, we can use triangular membership functions with the same interval.

Comparing with the true normal distribution, the uniform distribution would miss some data. For example, for the normally distributed input $X_1 \sim N(1, 0.05)$, suppose we use a uniformly distributed random variable on the interval [0.905960, 1.094040], resulting in 6% of data missing $[P(X_1 > 1.09494 \text{ or } X_1 < 0.90596) = 0.06]$. We can use similar uniform distributions for other random design variables and parameters.

The corresponding fuzzy variables have membership functions with the interval lengths of 0.188079 for $X_1 \sim X_7$; 0.02257 for $X_8$, $X_9$; and 37.62 for $X_{10}$, $X_{11}$, respectively. Using the proposed MPS method and SQP [25], the possibility-based optimal design is obtained as shown in Fig. 6, and Tables 9 and 10.

**Table 9   Possibility-based design optimization cost and design variable histories of side impact problem**

| Iter. | cost | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 30.83 | 1.000 | 1.000 | 1.000 | 1.000 | 2.000 | 1.000 | 1.000 | 0.300 | 0.300 |
| 1 | 32.55 | 1.040 | 1.350 | 0.950 | 1.500 | 1.534 | 1.200 | 0.400 | 0.345 | 0.192 |
| 2 | 32.67 | 1.030 | 1.350 | 0.958 | 1.500 | 1.598 | 1.200 | 0.400 | 0.345 | 0.192 |
| 3 | 32.64 | 1.027 | 1.350 | 0.955 | 1.500 | 1.601 | 1.200 | 0.400 | 0.345 | 0.192 |
| 4 | 32.63 | 1.027 | 1.350 | 0.954 | 1.500 | 1.600 | 1.200 | 0.400 | 0.345 | 0.192 |
| Opt. | 32.63 | 1.027 | 1.350 | 0.954 | 1.500 | 1.600 | 1.200 | 0.400 | 0.345 | 0.192 |

**Table 10  Possibility-based design optimization constraint history and number of function evaluations**

| Iter. | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $G_6$ | $G_7$ | $G_8$ | $G_9$ | $G_{10}$ | No. of Analyses |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.29 | −0.04 | 0.18 | −0.52 | −0.63 | −0.03 | 0.31 | 0.38 | −0.13 | −1.00 | 21 |
| 1 | 0.00 | −1.36 | −0.43 | −1.01 | −0.78 | −0.06 | 0.00 | −0.58 | 0.05 | −2.94 | 19 |
| 2 | 0.00 | −1.33 | −0.39 | −0.95 | −0.76 | −0.06 | 0.00 | −0.58 | 0.05 | −2.94 | 19 |
| 3 | 0.00 | −1.32 | −0.39 | −0.95 | −0.76 | −0.06 | 0.00 | −0.57 | 0.00 | −2.94 | 57 |
| 4 | 0.00 | −1.33 | −0.39 | −0.95 | −0.76 | −0.06 | 0.00 | −0.57 | 0.00 | −2.94 | 21 |
| Opt. | Act. | Inact. | Inact. | Inact. | Inact. | Inact. | Act. | Inact. | Act. | Inact. | 154 |

**Table 11  Probabilities of failure for components and system**

| Optimum designs | cost | $P(G_1 > 0)$ | $P(G_7 > 0)$ | $P(G_9 > 0)$ | $P$ (system failure) |
|---|---|---|---|---|---|
| PBDO | 32.63 | 0.00412 | 0.00149 | 0.00008 | 0.00297 |
| RBDO | 28.19 | 0.00763 | 0.02560 | 0.00309 | 0.03582 |

**Table 12  Probabilities of failure for components and system**

| Optimum designs | cost | $P(G_1 > 0)$ | $P(G_7 > 0)$ | $P(G_9 > 0)$ | $P$ (system failure) |
|---|---|---|---|---|---|
| PBDO | 29.05 | 0.00910 | 0.01277 | 0.00131 | 0.02258 |
| RBDO | 26.69 | 0.02743 | 0.07433 | 0.01255 | 0.11156 |

Using the uniform distributions on these intervals and RBDO, the optimum cost is obtained as 28.19 and the optimum design variables are [0.683, 1.350, 0.574, 1.500, 1.539, 1.200, 0.400, 0.345, 0.192]. Table 11 shows comparison of PBDO and RBDO results using Monte Carlo simulation. As shown in the table, the RBDO result does not satisfy the component level reliability target of 0.02275 for 7th constraint, while PBDO result satisfies both component level and system level reliability.

Next, consider the situation of 14% data missing, or fuzzy variables have membership functions with the interval length 0.14758 for $X_1 \sim X_7$; 0.01771 for $X_8$, $X_9$, and 29.52 for $X_{10}$, $X_{11}$, respectively. Using triangular membership functions on these intervals and PBDO, the optimum cost is 29.05 and the optimum design variables are [0.715, 1.350, 0.671, 1.500, 1.557, 1.200, 0.400, 0.345, 0.192]. Using uniform distributions on these intervals and RBDO, the optimum cost is 26.69 and the optimum design variables are [0.558, 1.350, 0.500, 1.432, 1.487, 1.200, 0.400, 0.345, 0.192]. Table 12 shows comparison of PBDO and RBDO result using Monte Carlo simulation. As shown in the table, the RBDO result does not satisfy the component level reliability for 1st and 7th constraints, while the PBDO result still satisfies both component level and system level reliability target of 0.02275.

## Discussions and Conclusion

When little information is available for the input data, the possibility-based method is better because it can assure a conservative possibilistic optimum design that is consistent with the limited information, which is a desirable merit, because more conservative optimum design is preferred when accurate statistical information is not available. It is also found that PBDO is more desirable than RBDO, because PBDO can inherently handle a system level possibility unlike RBDO.

This paper proposed membership function generation methods from the available limited data; and a maximal possibility search (MPS) method with interpolation to resolve disadvantages of vertex method and multilevel-cut method, such that it evaluates possibility constraints efficiently, stably, and accurately for nonlinear structural applications. In this paper, numerical results demonstrated that the proposed MPS method successfully performs inverse possibility analysis even for highly nonlinear and nonmonotonic problems.

Moreover, a new formulation of PBDO is proposed using the PMA. It is shown through several examples that it can carry out PBDO efficiently and accurately.

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